

Spiraling of adjacent trajectories in chaotic systems

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Abstract

The spiraling of adjacent trajectories in chaotic dynamical systems can be characterized by the distribution of local angular velocities of rotation of the displacement vector, which is governed by linearized equations of motion. This distribution, akin to that of local Lyapunov exponents, is studied for three examples of three-dimensional flows. Toy model shows that the rotation rate of adjacent trajectories influences on the rate of mixing of dynamic variables and on the sensitivity of trajectories to perturbations.

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I. INTRODUCTION

In nonlinear dynamics of systems - flows with equations of motion

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad (1)$$

where $\mathbf{x}(t)$ is a vector representing the state of the system in a K -dimensional phase space, $\mathbf{x} = \{x_i\}$, $1 \leq i \leq K$, definitive role is played by the properties of evolution of the displacement vector $\mathbf{r}(t)$ that is governed by the linear system of equations

$$\dot{\mathbf{r}} = \hat{M}\mathbf{r}. \quad (2)$$

Here \hat{M} is the local stability matrix with the elements $M_{ij} = \partial F_i / \partial x_j$, that are taken at the points of the phase trajectory $\mathbf{x}(t)$ and depend, generally, on time. The existence of the exponentially growing solutions of the system (2) serves as a definition of chaoticity of motion of the system (1).

For almost any initial conditions the displacement vector $\mathbf{r}(t)$ will approach the eigenmode of the system (2) that has the largest rate of the exponential growth, given by the first (maximal) characteristic Lyapunov exponent σ_1 or simply the Lyapunov exponent σ [1, 2]. Thus the behaviour of $\mathbf{r}(t)$ at large times $t \gg \sigma^{-1}$ loses its dependence on the initial conditions and represents some characteristics of the chaotic component as a whole. This property serves as a base for the standard method of numerical calculation of the Lyapunov exponent [3], that is determined from the rate of growth of the length of vector $\mathbf{r}(t)$. However, the evolution of the orientation of this vector is usually ignored.

We restrict ourselves by the phase space with dimensionality $K = 3$, the minimal one in which the chaotic motion of systems - flows like (1) is possible. Then the evolution of the direction of the displacement vector $\mathbf{r}(t)$ can be described as a rotation around the direction of the phase trajectory: the adjacent trajectories can be spiraling, as it is shown in figure 1.

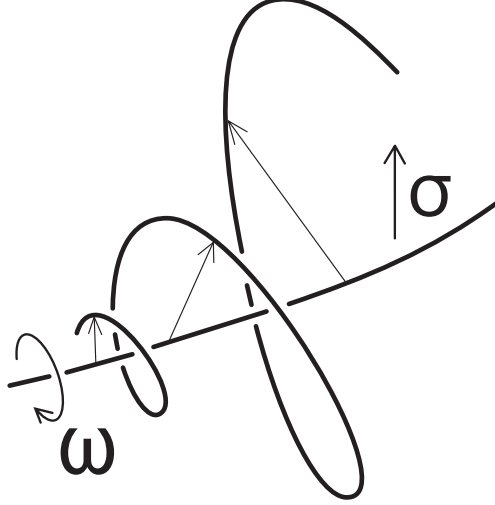


FIG. 1: Qualitative scheme of position of adjacent phase trajectories for chaotic motion in three-dimensional phase space. Whereas the displacement vector grows in length by the exponential law with the rate σ , it may also rotate with the angular velocity ω

The instantaneous angular velocity of this rotation $\omega(t)$ (we shall also call it "the rotation rate") is defined by the expression

$$\omega(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \cdot \frac{\mathbf{r}(t + \epsilon) \times \mathbf{r}(t) \cdot \mathbf{v}(t)}{\|\mathbf{r}(t + \epsilon)\| \|\mathbf{r}(t)\| \|\mathbf{v}(t)\|}, \quad (3)$$

where $\mathbf{v}(t) = \dot{\mathbf{x}}(t)$ is a vector of phase velocity, that is tangent to the phase trajectory, and $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ denotes the mixed product of three vectors.

The distribution W of values of $\omega(t)$ does not depend on the choice of the initial moment of time or the initial conditions for the system (2) as far as they belong to the same chaotic component. It is as universal as the distribution of the local Lyapunov exponents that are defined by the expression

$$\sigma(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \ln \frac{\|\mathbf{r}(t + \epsilon)\|}{\|\mathbf{r}(t)\|}. \quad (4)$$

The latter quantity is widely used for studies of the structure of the phase space of chaotic systems [4, 5, 6]. From comparison of equations (3) and (4) one can assume that the instantaneous angular velocity in Pickwickian sense can be considered as an imaginary part of the local Lyapunov exponent and could be used as a probe for exploration of fine details of chaotic dynamics.

II. THREE EXAMPLES

The distribution $W(\omega)$ could be easily obtained numerically. As the first example we consider the famous Lorenz model [7] given by the equations of motion

$$\begin{aligned}\dot{X} &= -\sigma X + \sigma Y, \\ \dot{Y} &= -XZ + rX - Y, \\ \dot{Z} &= XY - bZ,\end{aligned}\tag{5}$$

with the standard values of parameters $\sigma = 10$, $r = 28$, $b = 8/3$. The distribution $W(\omega)$ is shown in figure 2.

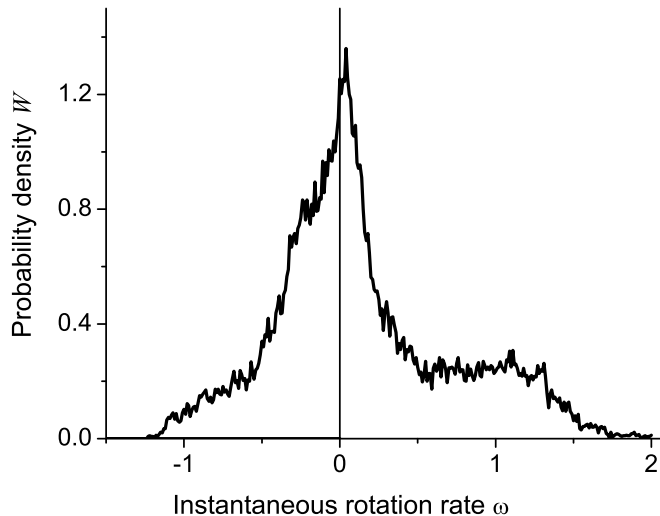


FIG. 2: Distribution W of the instantaneous rotation rate ω for the Lorenz model defined by the system of equations (5) with the standard values of parameters $\sigma = 10$, $r = 28$, $b = 8/3$.

The average value of the rate of rotation $\bar{\omega} = 0.158$ is much less than the width of the distribution (standard deviation of ω) $\Delta\omega = 0.63$. It may be noted that for the Lorenz model with the used values of parameters the characteristic frequency of motion Ω estimated from the power spectrum of the variable $Z(t)$ is about $\Omega \approx 8$, whereas the Lyapunov exponent $\sigma \approx 1.0$. Thus the small value of $\bar{\omega}$ may indicate the presence of the additional time scale of motion. We also note that the time of decay of correlations of $\omega(t)$ is rather close to $\tau \approx \Omega^{-1}$.

For the second example we take a conservative autonomous system, the Pullen - Edmonds

model oscillator [8] with the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2} + U(x, y) = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} (x^2 + y^2 + x^2 y^2), \quad (6)$$

where x, y are the Cartesian coordinates of the particle and p_x, p_y are the corresponding canonically conjugated momenta. In expression (6) the particle mass m , the frequency of small oscillations ω_0 and the nonlinearity length λ have been used as the unit scales. The motion of this system takes place on a three-dimensional energy surface in the four-dimensional phase space. Since the energy of the system is conserved, $E \equiv H = \text{const}$, the absolute value of the total momentum $p(x, y) = \sqrt{2[E - U(x, y)]}$ is completely defined by the position of the phase point in the configurational space. The angle φ that gives the direction of the total momentum is defined by the equations

$$p_x = p \cos \varphi, \quad p_y = p \sin \varphi. \quad (7)$$

It can be taken as the third dynamic variable, that yields the equations of motion

$$\dot{x} = p \cos \varphi, \quad \dot{y} = p \sin \varphi, \quad \dot{\varphi} = \frac{1}{p} \left(\frac{\partial U}{\partial x} \sin \varphi - \frac{\partial U}{\partial y} \cos \varphi \right). \quad (8)$$

For the motion on the energy surface $E = 15$, that is nearly ergodic [9], the distribution $W(\omega)$ is shown in figure 3.

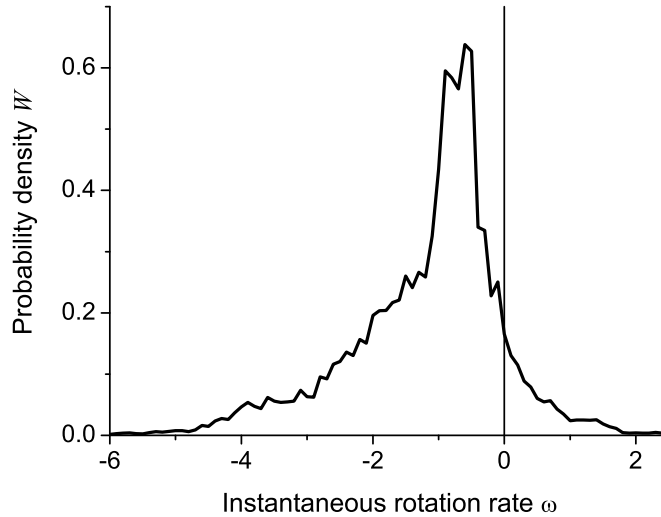


FIG. 3: Distribution W of the instantaneous rotation rate ω for the Pullen - Edmonds model, defined by the system of equations (8), on the energy surface $E = 15$.

The average value of the rate of rotation $\overline{\omega} = -1.49$ is comparable to the width of the distribution $\Delta\omega = 1.53$ and to the characteristic frequency of motion $\Omega \sim 1$.

For the third example we take a conservative non-autonomous system with the equation of motion

$$\ddot{x} + x^3 = F \cos \Omega t, \quad (9)$$

that describes an oscillator with cubic nonlinearity under the influence of external harmonic force (Duffing model). The system (9) could be turned into an autonomous three-dimensional system by introducing dynamical variables x , $y \equiv \dot{x}$ and interpreting the time t in the RHS as a third dynamic variable z that obeys the equation of motion $\dot{z} = 1$ [1]. For the motion in the chaotic component that surrounds the line $x = 0$, $y = 0$ the distribution $W(\omega)$ is shown in figure 4.

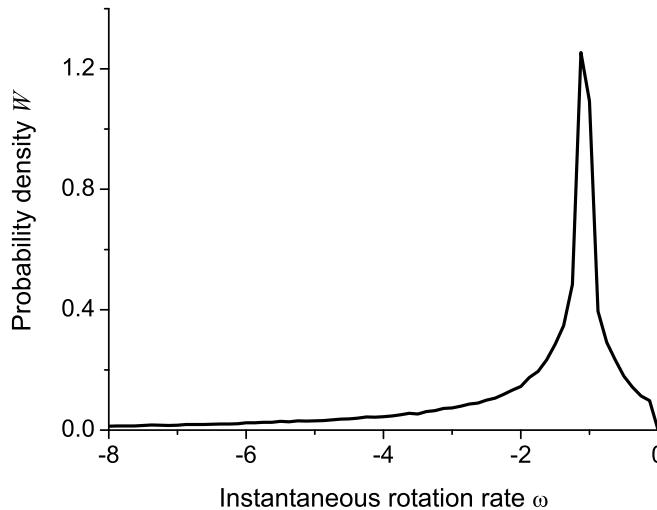


FIG. 4: Distribution W of the instantaneous rotation rate ω for the Duffing model defined by equation (9) with parameters $F = 5$, $\Omega = 1$.

It follows from equation (9) that the rotation rate is always negative. Its average value $\bar{\omega} = -2.24$ is comparable to the width of the distribution $\Delta\omega = 2.38$. We note that the value of $\bar{\omega}$ is not trivial: it is incommensurate with the driving frequency $\Omega = 1$.

From comparison of graphs in figures 2 - 4 it may be noticed that although our examples were taken from very different classes of dynamical systems, the forms of distributions have some common features: they are asymmetric, strongly peaked at a single non-zero most probable value and have extended wings.

III. SPIRALING AND MIXING

The proof of usefulness of the quantity $\omega(t)$ for the exploration of the phase space we leave for the future studies. There is another way to reveal the relevance of $\omega(t)$ to the properties of chaotic motion: to find a relation between this quantity and some established characteristics of chaoticity. The affinity between $\overline{\omega}$ and σ suggests that relations that include the Lyapunov exponent may be extended to include ω in a similar way.

In nonlinear dynamics there is a widely known rule of thumb stating that the rate of mixing (the inverse time of damping of correlations of dynamic variables) γ is approximately equal to the Lyapunov exponent:

$$\gamma \approx \sigma. \quad (10)$$

This relation sometimes turns into exact equality (*e.g.* for the linear random number generator $x' = \{Qx\}$, where $\{\cdot\}$ denotes the fractional part of a number, with integer parameter Q); sometimes equation (10) reflects the scaling property $\gamma \propto \sigma$ (*e.g.* it holds exactly for billiards), and in general it is expected to be fulfilled at least semiquantitatively for arbitrary systems [10]. From the picture in figure 1 one could surmise that the increase of ω will intensify the rate of exploration of the phase space and thus enhance the mixing.

For the check of this hypothesis we retreat to two-dimensional mappings, that can be interpreted as Poincare mappings for three-dimensional flows. In this class it is easy to construct a model that permits controllable and independent changes of ω and σ . Let $\{\xi_n, \eta_n\}$ be the components of the vector \mathbf{s}_n of normalized displacement at the n -th iteration, $\mathbf{s}_n = \mathbf{r}_n / \|\mathbf{r}_n\|$. Then the rate of rotation is equal to the angle of rotation for one step:

$$\omega_{n+1} = \arcsin(\xi_{n+1}\eta_n - \xi_n\eta_{n+1}). \quad (11)$$

Let's define the rate of damping of correlations (rate of mixing) of a dynamical variable with the known correlation function $B(n)$ by the relation

$$\gamma = \ln \left(\frac{S}{S-1} \right), \quad S = \frac{1}{B(0)} \sum_{i=0}^{\infty} |B(i)| \quad (12)$$

For the exponentially decreasing correlation function $B(n) = B(0) \exp(-\alpha n)$ this definition produces the rate of mixing equal to the exponent, $\gamma = \alpha$.

For the study of influence of rotation on mixing we take a two-dimensional mapping for which it is possible to variate rotation rate ω with constant σ , namely a piecewise linear

mapping of a unit square on itself

$$x' = \{ax + by\}, \quad y' = \{-bx + ay\}, \quad (13)$$

where

$$a = \exp \sigma \cos \omega, \quad b = \exp \sigma \sin \omega. \quad (14)$$

The behaviour of rates of mixing for dynamical variables x and y is shown in figure 5.

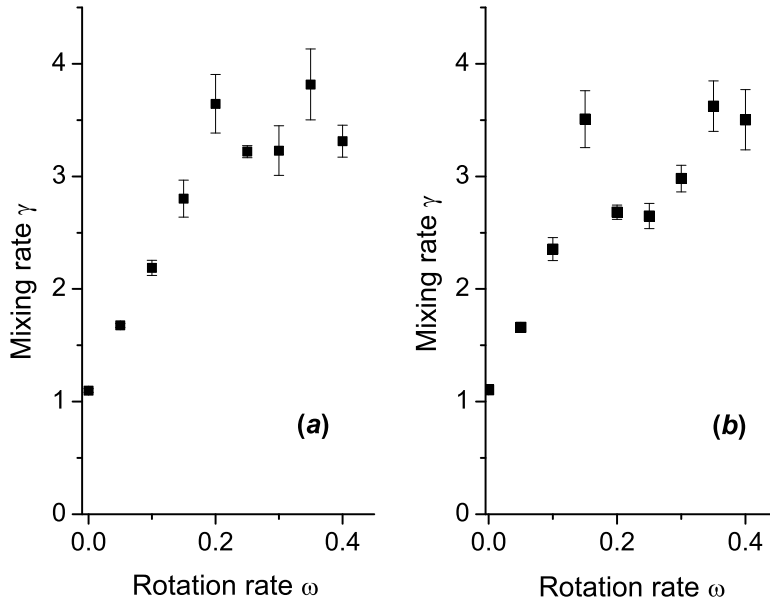


FIG. 5: Dependence of the mixing rate γ on the rotation rate ω for the two-dimensional mapping (13): (a) - for variable x ; (b) - for variable y .

The results of numerical calculation confirms the intuitive guess: at small values of ω its increase leads to the increase of the rate of mixing γ .

IV. SPIRALING AND RESPONSE TO PERTURBATIONS

Let's compare the system (1) to its perturbed version with the equations of motion

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \varepsilon \mathbf{V}(\mathbf{x}). \quad (15)$$

where $\varepsilon \mathbf{V}(\mathbf{x})$ is some static perturbation. Let $\mathbf{x}_0(t)$ and $\mathbf{x}_V(t)$ be the laws of motion of the unperturbed system (1) and perturbed system (15) correspondingly with the same initial conditions, and let $\Delta(t) = |\mathbf{x}_0(t) - \mathbf{x}_V(t)|$ denote the distance between the phase trajectories at a given moment t . For a given point of the phase space let's define a quantity

$$\Delta(\mathbf{x}) = \lim_{\varepsilon \rightarrow 0, t \rightarrow \infty} \frac{\Delta(t)}{|\varepsilon|} \exp(-\sigma t), \quad (16)$$

that defines the amplitude of the response of the system to the perturbation of a given form. This quantity, averaged over a given chaotic component of the phase space, $\overline{\Delta} = \langle \Delta(\mathbf{x}) \rangle$, gives a convenient measure of sensitivity of the system (1) to small perturbations.

Let's treat the simplest model of the evolution of deviations given by a system of flows

$$\dot{\xi} - \sigma\xi + \omega\eta = U, \quad \dot{\eta} - \omega\xi - \sigma\eta = V, \quad (17)$$

with constant parameters σ , ω , U and V . In the absence of perturbations ($U = V = 0$) it describes the rotation of the displacement vector with the constant angular velocity ω and its exponential growth with the exponent σ . For $U, V \neq 0$ the asymptotics of solutions for $\sigma t \gg 1$ have the form

$$\begin{aligned} \xi &\approx \frac{e^{\sigma t}}{\sigma^2 + \omega^2} [(\sigma U - \omega V) \cos \omega t + (\omega U + \sigma V) \sin \omega t], \\ \eta &\approx \frac{e^{\sigma t}}{\sigma^2 + \omega^2} [(\omega U + \sigma V) \cos \omega t + (-\sigma U + \omega V) \sin \omega t]. \end{aligned} \quad (18)$$

By definition (16) for each of the perturbations we have

$$\overline{\Delta} = \frac{1}{\sqrt{2(\sigma^2 + \omega^2)}}. \quad (19)$$

Therefore the schematic model (17) shows that with the increase of the rotation rate the amplitude of the response will decrease. One may infer that this connection may be universal.

Let's return to the piecewise model (13) perturbed by small variations of its control parameters:

$$x' = \{(a + \varepsilon)x + by\}, \quad y' = \{-bx + ay\} \quad (20)$$

and

$$x' = \{ax + (b + \varepsilon)y\}, \quad y' = \{-bx + ay\}. \quad (21)$$

Fig. 6 demonstrates that with the increase of the rotation rate ω the averaged amplitude of response to the perturbation $\overline{\Delta}$ in general decreases, albeit non-monotonously, in analogy with the dependence (19).

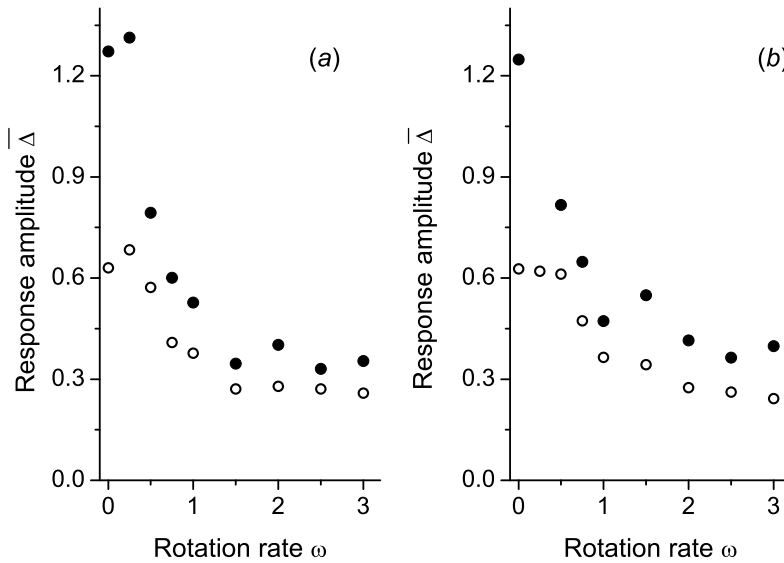


FIG. 6: Dependence of the response amplitude $\bar{\Delta}$ on the rotation rate ω for different perturbations of the two-dimensional mapping (13): (a) - perturbed system is given by equation (20); (b) - perturbed system is given by equation (21). In both pictures filled circles are for $\sigma = 0.25$ and open circles - for $\sigma = 0.5$.

Recently the changes of the evolution under the influence of small perturbation have attracted considerable attention in the theory of Hamiltonian chaotic systems in connection with the problem of dynamics of fidelity [11, 12, 13, 14]. The example described in this section shows that the rotation rate can display itself in the preexponential factors of this response.

V. GENERALIZATION TO HIGHER DIMENSIONS

Everywhere above we have restricted ourselves to analysis of trajectories in three-dimensional phase space. The generalization of the definition (3) to higher dimensions could be obtained straightforwardly from the expression for the oriented volume V_K in K -dimensional space [15]. It can be constructed from the unit vector along the direction of phase velocity $\mathbf{u}(t) = \mathbf{v}(t)/\|\mathbf{v}(t)\|$ and $(K - 1)$ unit vectors along the displacement vectors taken in consequent equidistant moments of time, $\mathbf{s}(t + k\epsilon) = \mathbf{r}(t + k\epsilon)/\|\mathbf{r}(t + k\epsilon)\|$, $1 \leq k \leq K - 2$ by their convolution with the completely antisymmetric tensor of the K -th rank $\mathbf{E}_{ijk\dots n}$:

$$V_K(t) = E_{ijk\dots n} u_i s_j(t) s_k(t + \epsilon) \dots s_n(t + (K - 2)\epsilon). \quad (22)$$

However, this quantity scales as ϵ^N , where $N = (K - 1)(K - 2)/2$, and the generalized rate of "rotation",

$$\omega_K(t) = \lim_{\epsilon \rightarrow 0} \epsilon^{-N} V_K(t), \quad (23)$$

for $K > 3$ depends not only on the elements of stability matrix at a given moment of time but also on their time derivatives up to the order $N - 1$.

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